Particle approximation of the intensity measures of a spatial branching point process arising in multi-target tracking

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Abstract

The aim of this paper is two-fold. First we analyze the sequence of intensity measures of a spatial branching point process arising in a multiple target tracking context. We study its stability properties, characterize its long time behavior and provide a series of weak Lipschitz type functional contraction inequalities. Second we design and analyze an original particle scheme to approximate numerically these intensity measures. Under appropriate regularity conditions, we obtain uniform and non asymptotic estimates and a functional central limit theorem. To the best of our knowledge, these are the first sharp theoretical results available for this class of spatial branching point processes.

Keywords : Spatial branching processes, multi-target tracking problems, mean field and interacting particle systems, w.r.t. time, functional central limit theorems.

1   Introduction

Multi-target tracking problems deal with tracking several targets simultaneously given noisy sensor measurements. Over recent years, point processes approaches to address these problems have become very popular. The use of point processes in a multiple-target tracking context was first proposed in S. Mori et al. [15] as early as in 1986. Using a random sets formalism, a formalism essentially equivalent to the point process formalism [16], R. Mahler and his co-authors proposed in two books [11, 12] a systematic treatment of multi-sensor multi-target filtering problems. However, as mentioned in [16], "... although the random sets formalism (or the point process formalism) for multitarget tracking has provided a unified view on the subject of multiple target tracking, it has failed to produce any significant practical tracking algorithms...".

This situation has recently changed following the introduction of the PHD (probability hypothesis density) filter by R. Mahler [13, 14]. The PHD filter is a powerful multi-target tracking algorithm which is essentially a Poisson type approximation to the optimal multi-target filter [13, 14, 17]. It has found numerous applications since its introduction. The PHD filter cannot be computed analytically but it can be approximated by a mixture of Gaussians for linear Gaussian target models [18] and by non-standard particle methods for nonlinear non-Gaussian target models [9, 10].

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Despite their increasing popularity, the theoretical performance of these multi-target particle methods remain poorly understood. Indeed their mathematical structure is significantly different from standard particle filters so the detailed theoretical results for particle filters provided in [4] are not applicable. Some convergence results have already been established in [9, 10] but remain quite limited. Reference [9] presents a basic convergence result for the PHD filter but does not establish any rate of convergence. In [10] the authors provide some quantitative bounds and a central limit theorem. However these quantitative bounds are not sharp and no stability result is provided.

The aim of this work is to initiate a thorough theoretical study of these non-standard particle methods by first characterizing the stability properties of the "signal" process and establishing uniform w.r.t. the time index convergence results for its particle approximation. This "signal" process is a spatial branching point process whose intensity measure always satisfies a closed recursive equation in the space of bounded positive measures. We will not consider any observation process in this article. The analysis of the particle approximations of PHD filters is presented in [7]. It builds heavily upon the present work but is even more complex as it additionally involves at each time step a nonlinear update of the intensity measure.

The rest of this paper is organized as follows:

In section 2, we present a spatial branching point process which is general enough to model a wide variety of multiple target problems. We establish the linear evolution equation associated to the intensity measures of this process and introduce an original particle scheme to approximate them numerically. Section 3 summarizes the main results of this paper. In Section 4, we provide a detailed analysis of the stability properties and the long time behavior of this sequence of intensity measures, including the asymptotic behavior of the total mass process, i.e. the integral of the intensity measure over the state space, and the convergence to equilibrium of the corresponding sequence of normalized intensity measures. For time-homogeneous models, we exhibit three different types of asymptotic behavior. The analysis of these stability properties is essential in order to guarantee the robustness of the model and to obtain reliable numerical approximation schemes. Section 5 is devoted to the theoretical study of the non-standard particle scheme introduced to approximate the intensity measures. Our main result in this section is a non-asymptotic convergence for this scheme. Under some appropriate stability conditions, we additionally obtain uniform estimates w.r.t. the time parameter.

2 Spatial branching point process and its particle approximation

2.1 Spatial branching point process for multi-target tracking

Assume that at a given time n there are N_n target states $(X_n^i)_{1 \le i \le N_n}$ taking values in some measurable state space E_n enlarged with an auxiliary cemetery point c. The state space E_n depends on the problem at hand. It may vary with the time parameter and can include all the characteristics of a target such as its type, its kinetic parameters as well as its complete path from the origin. As usual, we extend the measures γ_n and the bounded measurable functions f_n on E_n by setting $\gamma_n(c) = 0$ and $f_n(c) = 0$.

Each target has a survival probability $e_n(X_n^i) \in [0, 1]$. When a target dies, it goes to the cemetery point c. We also use the convention $e_n(c) = 0$ so that a dead target can only stay in the cemetery. Survival targets give birth to a random strictly positive number of individuals $h_n^i(X_n^i)$ where $\left(h_n^i(X_n^i)\right)_{1 \le i \le N_n}$ is a collection of independent random variables such that $\mathbb{E}\left(h_n^i(x_n)\right) = H_n(x_n)$ for any $x_n \in E_n$ where H_n is a given collection of bounded

functions H_n. We have $H_n(x_n) \geq 1$ for any $x_n \in E_n$ as $h_n^i(x_n) \geq 1$. This branching transition is called spawning in the multi-target tracking literature. We define $G_n = e_n H_n$.

After this branching transition, the system consists of a random number $\widehat{N}_n$ of individuals $(\widehat{X}_n^i)_{1 \leq i \leq \widehat{N}_n}$. Each of them evolves randomly $\widehat{X}_n^i = x_n \rightsquigarrow X_{n+1}^i$ according to a Markov transition $M_{n+1}(x_n, dx_{n+1})$ from E_n into E_{n+1}. We use the convention $M_{n+1}(c, c) = 1$, so that any dead target remains in the cemetery state.

At the same time, an independent collection of new targets is added to the current configuration. This additional point process is modeled by a spatial Poisson process with a prescribed intensity measure μ_{n+1} on E_{n+1}. It is used to model new targets entering the state space.

At the end of this transition, we obtain $N_{n+1} = \widehat{N}_n + N'_{n+1}$ targets $(X_{n+1}^i)_{1 \leq i \leq N_{n+1}}$, where N'_{n+1} is a Poisson random variable with parameter given by the total mass $\mu_{n+1}(1)$ of the positive measure μ_{n+1}, and $(X_{n+1}^{\widehat{N}_n+i})_{1 \leq i \leq N'_{n+1}}$ are independent and identically distributed random variables with common distribution $\overline{\mu}_{n+1} = \mu_{n+1}/\mu_{n+1}(1)$ where $\mu_{n+1}(1) := \int_{E_{n+1}} \mu_{n+1}(dx)$.

Example. *To illustrate the model, we present here a simple yet standard example [18] of a target evolving in a two-dimensional surveillance region $S \subset \mathbb{R}^2$. In this case, we set $E_n = E = S \times \mathbb{R}^2$. All the targets are assumed to be of the same type. The state of a target $X_n = [p_n^x, p_n^y, v_n^x, v_n^y]^T$ consists of its position $(p_n^x, p_n^y) \in S$ and velocity $(v_n^x, v_n^y) \in \mathbb{R}^2$ and is assumed to evolve according to a linear Gaussian model*

$$X_n = AX_{n-1} + V_n \tag{2.1}$$

where A is a known transition matrix and $V_n \sim \mathcal{N}(0, \Sigma)$ is a sequence of i.i.d zero-mean normal random variables of covariance Σ; i.e. $M_n(x_{n-1}, dx_n) = M(x_{n-1}, x_n) dx_n$ with

$$M(x_{n-1}, x_n) = |2\pi\Sigma|^{-1/2} \exp\left(-\frac{1}{2}(x_n - Ax_{n-1})^T \Sigma^{-1}(x_n - Ax_{n-1})\right).$$

In the example, we assume that $\mu_n(x) = \mu(x)$, $e_n(x) = s > 0$ and $h_n(x_n) = h \in \{1, 2\}$ with $\mathbb{P}(h = 1) = 1 - \mathbb{P}(h = 2) = \alpha$. Hence for this model, each target X_{n-1} survives at time $n - 1$ with a probability s. Each survival target has one offspring with probability α which evolves according to (2.1) or two offspring with probability $1 - \alpha$ which, conditional upon X_{n-1}, independently evolve according to (2.1). Additionally, a random number of targets distributed according to a Poisson distribution of parameter $\mu(1)$ appear. These targets are independent and distributed in E as $\overline{\mu} = \mu/\mu(1)$.

2.2 Sequence of intensity distributions

At every time n, the intensity measure of the point process $\mathcal{X}_n := \sum_{i=1}^{N_n} \delta_{X_n^i}$ associated to the targets is given for any bounded measurable function f on $E_n \cup \{c\}$ by the following formula:

$$\gamma_n(f) := \mathbb{E}(\mathcal{X}_n(f)) \quad \text{with} \quad \mathcal{X}_n(f) := \int f(x)\,\mathcal{X}_n(dx)$$

To simplify the presentation, we suppose that the initial configuration of the targets is a spatial Poisson process with intensity measure μ_0 on the state space E_0.

Given the construction defined in section 2.1, it follows straightforwardly that the intensity measures γ_n on E_n satisfy the following recursive equation.

Lemma 2.1 *For any $n \geq 0$, we have*

$$\gamma_{n+1}(dx') = \int \gamma_n(dx)\,Q_{n+1}(x, dx') + \mu_{n+1}(dx') \tag{2.2}$$

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with the initial condition $\gamma_0 = \mu_0$ where μ_{n+1} is the intensity measure of the spatial point process associated to the birth of new targets at time $n+1$ while the integral operator Q_{n+1} from E_n into E_{n+1} is defined by

$$Q_{n+1}(x_n, dx_{n+1}) := G_n(x_n) \ M_{n+1}(x_n, dx_{n+1}). \tag{2.3}$$

Proof:

For any bounded measurable function f on $E_{n+1} \cup \{c\}$, we have

$$\gamma_{n+1}(f) = \mathbb{E}\left(\sum_{i=1}^{\widehat{N}_n} f\left(X_{n+1}^i\right)\right) + \mathbb{E}\left(\sum_{i=\widehat{N}_n}^{\widehat{N}_n + N'_{n+1}} f\left(X_{n+1}^i\right)\right)$$

where, thanks to the Poisson assumption, we have

$$\mathbb{E}\left(\sum_{i=\widehat{N}_n}^{\widehat{N}_n + N'_{n+1}} f\left(X_{n+1}^i\right)\right) = \mu_{n+1}(1)\overline{\mu}_{n+1}(f) = \mu_{n+1}(f)$$

and

$$\mathbb{E}\left(\sum_{i=1}^{\widehat{N}_n} f\left(X_{n+1}^i\right)\right) = \mathbb{E}\left(\mathbb{E}\left(\sum_{i=1}^{\widehat{N}_n} f\left(X_{n+1}^i\right) \middle| \mathcal{F}_n\right)\right)$$

$$= \mathbb{E}\left(\mathbb{E}\left(\sum_{i=1}^{\widehat{N}_n} M_{n+1}(f)\left(\widehat{X}_n^i\right) \middle| \mathcal{G}_n\right)\right)$$

$$= \mathbb{E}\left(\sum_{i=1}^{N_n} e_n\left(X_n^i\right) H_n(X_n^i) M_{n+1}(f)\left(X_n^i\right)\right)$$

$$= \gamma_n\left(e_n H_n M_{n+1}(f)\right)$$

where $\mathcal{F}_n$ denotes the σ-field generated by $(\widehat{X}_n^i)_{1 \leq i \leq \widehat{N}_n}$ and $\mathcal{G}_n$ the σ-field generated by $(X_n^i)_{1 \leq i \leq N_n}$. ∎

These intensity measures typically do not admit any closed-form expression. A natural way to approximate them numerically is to use a particle interpretation of the associated sequence of probability distributions given by

$$\eta_n(dx) := \gamma_n(dx_n)/\gamma_n(1) \quad \text{with} \quad \gamma_n(1) := \int_{E_n} \gamma_n(dx)$$

To avoid unnecessary technical details, we further assume that the potential functions G_n are chosen so that for any $x \in E_n$

$$0 < g_{n,-} \leq G_n(x) \leq g_{n,+} < \infty \tag{2.4}$$

for any time parameter $n \geq 0$. Note that this assumption is satisfied in most realistic multi-target scenarios such as the example discussed at the end of section 2.1. Indeed the condition $g_{n,-} \leq G_n(x)$ essentially states that there exists $e_{n,-} > 0$ such that $e_n(x) \geq e_{n,-}$ for any $x \in E_n$ as $H_n(x) \geq 1$. The condition $G_n(x) \leq g_{n,+}$ states that there exists $H_{n,+} < \infty$ such that $H_n(x) \leq H_{n,+}$ for any $x \in E_n$ as $e_n(x) \leq 1$. In the unlikely scenario where (2.4) is

not satisfied then the forthcoming analysis can be extended to more general models using the techniques developed in section 4.4 in [4]; see also [3]. We denote by $\mathcal{P}(E_n)$ the set of probability measures on the state space E_n.

To describe these particle approximations, it is important to observe that the pair process $(\gamma_n(1), \eta_n) \in (\mathbb{R}_+ \times \mathcal{P}(E_n))$ satisfies an evolution equation of the following form

$$(\gamma_n(1), \eta_n) = \Gamma_n(\gamma_{n-1}(1), \eta_{n-1}) \tag{2.5}$$

We let Γ_n^1 and Γ_n^2 be the first and the second component mappings from $(\mathbb{R}_+ \times \mathcal{P}(E_n))$ into $\mathbb{R}_+$, and from $(\mathbb{R}_+ \times \mathcal{P}(E_n))$ into $\mathcal{P}(E_n)$. The mean field particle approximation associated with the equation (2.5) relies on the fact that it is possible to rewrite the mapping Γ_{n+1}^2 in the following form

$$\Gamma_{n+1}^2(\gamma_n(1), \eta_n) = \eta_n K_{n+1,(\gamma_n(1),\eta_n)} \tag{2.6}$$

where $K_{n+1,(m,\eta)}$ is a Markov kernel indexed by the time parameter n, a mass parameter $m \in \mathbb{R}_+$ and a probability measure η on the space E_n. In the literature on mean field particle systems, $K_{n,(m,\eta)}$ is called a McKean transition. The choice of such Markov transitions $K_{n,(m,\eta)}$ is not unique and will be discussed in section 5.1.

Before concluding this section, we note that

$$\gamma_{n+1}(dx') = (\gamma_n Q_{n+1})(dx') := \int \gamma_n(dx) \, Q_{n+1}(x, dx') \tag{2.7}$$

when $\mu_n = 0$. In this particular situation, the solution of the equation (2.2) is given by the following Feynman-Kac path integral formulae

$$\gamma_n(f) = \gamma_0(1) \, \mathbb{E} \left(f(X_n) \prod_{0 \leq p < n} G_p(X_p) \right) \tag{2.8}$$

where X_n stands for a Markov chain taking values in the state spaces E_n with initial distribution $\eta_0 = \gamma_0/\gamma_0(1)$ and Markov transitions M_n (see for instance section 1.4.4.in [4]). These measure-valued equations have been studied at length in [4].

2.3   Mean field particle interpretation

The transport formula presented in (2.6) provides a natural interpretation of the probability distributions η_n as the laws of a process $\overline{X}_n$ whose elementary transitions $\overline{X}_n \rightsquigarrow \overline{X}_{n+1}$ depends on the distribution $\eta_n = \mathrm{Law}(\overline{X}_n)$ as well as on the current mass $\gamma_n(1)$. In contrast to the more traditional McKean type nonlinear Markov chains presented in [4], the dependency on the mass process induces a dependency of the whole sequence of measures η_p, from the origin $p = 0$ up to the current time $p = n$.

From now on, we will always assume that the mappings

$$\left(m, \left(x^i \right)_{1 \leq i \leq N} \right) \in \left(\mathbb{R}_+ \times E_n^N \right) \mapsto K_{n+1,\left(m, \frac{1}{N} \sum_{i=1}^N \delta_{x^i} \right)} (x, A_{n+1})$$

are measurable w.r.t. the product sigma fields on $(\mathbb{R}_+ \times E_n^N)$, for any $n \geq 0$, $N \geq 1$, and $1 \leq i \leq N$, and any measurable subset $A_{n+1} \subset E_{n+1}$. In this situation, the mean field particle interpretation of (2.6) is an E_n^N-valued sequence $\xi_n^{(N)} = \left(\xi_n^{(N,i)} \right)_{1 \leq i \leq N}$ defined as

$$\begin{cases} \gamma_{n+1}^N(1) & = & \gamma_n^N(1) \, \eta_n^N(G_n) + \mu_{n+1}(1) \\ \\ \mathbb{P}\left(\xi_{n+1}^{(N)} \in dx \mid \mathcal{F}_n^{(N)} \right) & = & \prod_{i=1}^N K_{n+1,(\gamma_n^N(1),\eta_n^N)}(\xi_n^{(N,i)}, dx^i) \end{cases} \tag{2.9}$$

with the pair of occupation measures $\left(\gamma_n^N, \eta_n^N\right)$ defined below

$$\eta_n^N := \frac{1}{N} \sum_{i=1}^N \delta_{\xi_n^{(N,i)}} \quad \text{and} \quad \gamma_n^N(dx) := \gamma_n^N(1) \; \eta_n^N(dx)$$

In the above displayed formula, $\mathcal{F}_n^{(N)}$ stands for the σ-field generated by the random sequence $(\xi_p^{(N)})_{0 \leq p \leq n}$, and $dx = dx^1 \times \ldots \times dx^N$ stands for an infinitesimal neighborhood of a point $x = (x^1, \ldots, x^N) \in E_n^N$. The initial system $\xi_0^{(N)}$ consists of N independent and identically distributed random variables with common law η_0. As usual, to simplify the presentation, we will suppress the parameter N when there is no possible confusion, so that we write ξ_n and ξ_n^i instead of $\xi_n^{(N)}$ and $\xi_n^{(N,i)}$.

In the above discussion, we have implicitly assumed that the quantities $\mu_n(1)$ are known and that it is easy to sample from the probability distribution $\overline{\mu}_n(dx) := \mu_n(dx)/\mu_n(1)$. In practice, we often need to resort to an additional approximation scheme to approximate $\mu_n(1)$ and $\overline{\mu}_n$. This situation is discussed in section 6. This additional level of approximation has essentially a minimal impact on the properties of the particle approximation scheme which can be analyzed using the same tools.

2.4 Notation

For the convenience of the reader, we end this introduction with some notation used in the present article. We denote by $\mathcal{M}(E)$ the set of measures on some measurable state space $(E, \mathcal{E})$ and we recall that $\mathcal{P}(E)$ is the set of probability measures. We also denote $\mathcal{B}(E)$ the Banach space of all bounded and measurable functions f equipped with the uniform norm $\|f\|$ and $\mathrm{Osc}_1(E)$ the convex set of $\mathcal{E}$-measurable functions f with oscillations $\mathrm{osc}(f) \leq 1$ where $\mathrm{osc}(f) = \sup_{(x,y) \in E^2} |f(x) - f(y)|$.

We let $\mu(f) = \int \mu(dx) \, f(x)$ be the Lebesgue integral of a function $f \in \mathcal{B}(E)$ with respect to a measure $\mu \in \mathcal{M}(E)$. We recall that a bounded integral kernel $M(x, dy)$ from a measurable space $(E, \mathcal{E})$ into an auxiliary measurable space $(E', \mathcal{E}')$ is an operator $f \mapsto M(f)$ from $\mathcal{B}(E')$ into $\mathcal{B}(E)$ such that the functions $x \mapsto M(f)(x) := \int_{E'} M(x, dy) f(y)$ are $\mathcal{E}$-measurable and bounded for any $f \in \mathcal{B}(E')$. The kernel M also generates a dual operator $\mu \mapsto \mu M$ from $\mathcal{M}(E)$ into $\mathcal{M}(E')$ defined by $(\mu M)(f) := \mu(M(f))$. A Markov kernel is a positive and bounded integral operator M with $M(1)(x) = 1$ for any $x \in E$. Given a pair of bounded integral operators (M_1, M_2), we let $(M_1 M_2)$ be the composition operator defined by $(M_1 M_2)(f) = M_1(M_2(f))$. For time-homogenous state spaces, we denote by $M^k = M^{k-1} M = M M^{k-1}$ the k-th composition of a given bounded integral operator M, with $k \geq 0$, with the convention $M^0 = Id$ the identity operator. We also use the notation

$$M\left([f_1 - M(f_1)][f_2 - M(f_2)]\right)(x) := M\left([f_1 - M(f_1)(x)][f_2 - M(f_2)(x)]\right)(x)$$

for some bounded functions f_1, f_2.

We also denote the total variation norm on $\mathcal{M}(E)$ by $\|\mu\|_{\mathrm{tv}} = \sup_{f \in \mathrm{Osc}_1(E)} |\mu(f)|$. When the bounded integral operator M has a constant mass, that is $M(1)(x) = M(1)(y)$ for any $(x, y) \in E^2$, the operator $\mu \mapsto \mu M$ maps $\mathcal{M}(E)$ into $\mathcal{M}(E')$. In this situation, we let $\beta(M)$ be the Dobrushin coefficient of a bounded integral operator M defined by the following formula

$$\beta(M) := \sup \; \{\mathrm{osc}(M(f)) \; ; \; f \in \mathrm{Osc}_1(E)\}$$

Given a positive function G on E, we let $\Psi_G \; : \; \eta \in \mathcal{P}(E) \mapsto \Psi_G(\eta) \in \mathcal{P}(E)$ be the Boltzmann-Gibbs transformation defined by

$$\Psi_G(\eta)(dx) := \frac{1}{\eta(G)} \, G(x) \, \eta(dx)$$

We recall that $\Psi_G(\eta)$ can be expressed in terms of a Markov transport equation

$$\eta S_\eta = \Psi_G(\eta) \qquad (2.10)$$

for some selection type transition $S_\eta(x, dy)$. For instance, for any $\epsilon \geq 0$ s.t. $G(x) > \epsilon$ for any x, we notice that

$$\Psi_{(G-\epsilon)}(\eta) = \frac{\eta(G)}{\eta(G) - \epsilon} \left(\Psi_{(G)}(\eta) - \frac{\epsilon\eta}{\eta(G)} \right)$$

so we can take

$$S_\eta(x, dy) := \frac{\epsilon}{\eta(G)} \, \delta_x(dy) + \left(1 - \frac{\epsilon}{\eta(G)} \right) \, \Psi_{(G-\epsilon)}(\eta)(dy) \qquad (2.11)$$

For $\epsilon = 0$, we have $S_\eta(x, dy) = \Psi_G(\eta)(dy)$. We can also choose

$$S_\eta(x, dy) := \epsilon G(x) \, \delta_x(dy) + (1 - \epsilon G(x)) \, \Psi_G(\eta)(dy) \qquad (2.12)$$

for any $\epsilon \geq 0$ that may depend on the current measure η, and s.t. $\epsilon G(x) \leq 1$. For instance, we can choose $1/\epsilon$ to be the η-essential supremum of G.

3 Statement of the main results

At the end of section 2.2, we have seen that the evolution equation (2.2) coincides with that of a Feynman-Kac model (2.8) for $\mu_n = 0$. In this specific situation, the distributions γ_n are simply given by the recursive equation

$$\gamma_n = \gamma_{n-1} Q_n \implies \forall 0 \leq p \leq n \qquad \gamma_n = \gamma_p Q_{p,n} \quad \text{with} \quad Q_{p,n} = Q_{p+1} \ldots Q_{n-1} Q_n \qquad (3.1)$$

For $p = n$, we use the convention $Q_{n,n} = Id$. In addition, the nonlinear semigroup associated to this sequence of distributions is given by

$$\eta_n(f) = \Phi_{p,n}(\eta_p)(f) := \eta_p Q_{p,n}(f)/\eta_p Q_{p,n}(1) = \eta_p \left(Q_{p,n}(1) P_{p,n}(f) \right)/\eta_p Q_{p,n}(1) \qquad (3.2)$$

with the Markov kernel $P_{p,n}(x_p, dx_n) = Q_{p,n}(x_p, dx_n)/Q_{p,n}(x_p, E_n)$. The analysis of the mean field particle interpretations of such models has been studied in [4]. Various properties including contraction inequalities, fluctuations, large deviations and concentration properties have been developed for this class of models. In this context, the fluctuations properties as well as $\mathbb{L}_r$-mean error estimates, including uniform estimates w.r.t. the time parameter are often expressed in terms of two central parameters:

$$q_{p,n} = \sup_{x,y} \frac{Q_{p,n}(1)(x)}{Q_{p,n}(1)(y)} \quad \text{and} \quad \beta(P_{p,n}) = \sup_{x,y \in E_p} \|P_{p,n}(x, .) - P_{p,n}(y, .)\|_{\text{tv}} \qquad (3.3)$$

with the pair of Feynman-Kac semigroups $(P_{p,n}, Q_{p,n})$ introduced in (3.1) and (3.2).

We also consider the pair of parameters $(g_-(n), g_+(n))$ defined below

$$g_-(n) = \inf_{0 \leq p < n} \inf_{E_p} G_p \leq \sup_{0 \leq p < n} \sup_{E_p} G_p = g_+(n)$$

We also write $g_{-/+}(n)$ to refer to both parameters. The first main objective of this article is to extend some of these properties to models where μ_n is non necessarily null. We illustrate our estimates in three typical scenarios

$$\text{1)} \quad G = g_{-/+} = 1 \qquad \text{2)} \quad g_+ < 1 \quad \text{and} \quad \text{3)} \quad g_- > 1 \qquad (3.4)$$

arising in time homogeneous models

$$(E_n, G_n, M_n, \mu_n, g_-(n), g_+(n)) = (E, G, M, \mu, g_-, g_+) \tag{3.5}$$

These three scenarios correspond to the case where, *independently from the additional spontaneous births*, the existing targets die or survive and spawn in such a way that either their number remains constant ($G = g_{-/+} = 1$), decreases ($g_+ < 1$) or increases ($g_- > 1$).

Our first main result concerns three different types of long time behavior for these three types of models. This result can basically be stated as follows.

Theorem 3.1 *For time homogeneous models (3.5), the limiting behavior of $(\gamma_n(1), \eta_n)$ in the three scenarios (3.4) is as follows:*

1. *When $G(x) = 1$ for any $x \in E$, we have*

$$\gamma_n(1) = \gamma_0(1) + \mu(1)\, n \quad and \quad \|\eta_n - \eta_\infty\|_{\mathrm{tv}} = O\left(\frac{1}{n}\right)$$

   when M is chosen so that

$$\sum_{n \geq 0} \sup_{x \in E} \|M^n(x, .) - \eta_\infty\|_{\mathrm{tv}} < \infty \quad for\ some\ invariant\ measure\ \eta_\infty = \eta_\infty M. \tag{3.6}$$

2. *When $g_+ < 1$, there exists a constant $c < \infty$ such that*

$$\forall f \in \mathcal{B}(E), \qquad |\gamma_n(f) - \gamma_\infty(f)| \vee |\eta_n(f) - \eta_\infty(f)| \leq c\, g_+^n\, \|f\|$$

   with the limiting measures

$$\gamma_\infty(f) := \sum_{n \geq 0} \mu Q^n(f) \ and\ \eta_\infty(f) := \gamma_\infty(f)/\gamma_\infty(1) \tag{3.7}$$

3. *When $g_- > 1$ and there exist $k \geq 1$ and $\epsilon > 0$ such that $M^k(x, .) \geq \epsilon\, M^k(y, .)$ for any $x, y \in E$ then the mapping $\Phi = \Phi_{n-1,n}$ introduced in (3.2) has a unique fixed point $\eta_\infty = \Phi(\eta_\infty)$ and*

$$\lim_{n \to \infty} \frac{1}{n} \log \gamma_n(1) = \log \eta_\infty(G) \quad and \quad \|\eta_n - \eta_\infty\|_{\mathrm{tv}} \leq c\, e^{-\lambda n}$$

   for some finite constant $c < \infty$ and some $\lambda > 0$.

A more precise statement and a detailed proof of the above theorem can be found in section 4.2.

Our second main result concerns the convergence of the mean field particle approximations presented in (2.9). We provide rather sharp non asymptotic estimates including uniform convergence results w.r.t. the time parameter. Our results can be basically stated as follows.

Theorem 3.2 *For any $n \geq 0$, and any $N \geq 1$, we have $\gamma_n(1)$ and $\gamma_n^N(1) \in I_n$ with the compact interval I_n defined below*

$$I_n := [m_-(n), m_+(n)] \quad where \quad m_{-/+}(n) := \sum_{p=0}^{n} \mu_p(1) g_{-/+}(n)^{(n-p)} \tag{3.8}$$

In addition, for any $r \geq 1$, $f \in \mathrm{Osc}_1(E_n)$, and any $N \geq 1$, we have

$$\sqrt{N} \; \mathbb{E}\left(\left|\left[\eta_n^N - \eta_n\right](f)\right|^r\right)^{\frac{1}{r}} \leq a_r \; b_n \quad \text{with} \quad b_n \leq \sum_{p=0}^{n} b_{p,n} \tag{3.9}$$

where $a_r < \infty$ stands for a constant whose value only depends on the parameter r and $b_{p,n}$ is the collection of constants given by

$$b_{p,n} := 2 \; (1 \wedge m_{p,n}) \; q_{p,n} \left[q_{p,n} \; \beta(P_{p,n}) + \sum_{p<q\leq n} \frac{c_{q,n}}{\sum_{p<r\leq n} c_{r,n}} \; \beta(P_{q,n})\right] \tag{3.10}$$

with the pair of parameters

$$m_{p,n} = m_+(p)\|Q_{p,n}(1)\|/\sum_{p<q\leq n} c_{q,n} \quad \text{and} \quad c_{p,n} := \mu_p Q_{p,n}(1)$$

Furthermore, the particle measures γ_n^N are unbiased, and for the three scenarios (3.4) with time homogenous models s.t. $M^k(x, \cdot) \geq \epsilon M^k(y, \cdot)$, for any $x, y \in E$ and some pair of parameters $k \geq 1$ and $\epsilon > 0$, the constant b_n in (3.9) can be chosen so that $\sup_{n\geq 0} b_n < \infty$; in addition, we have the non asymptotic variance estimates for some $d < \infty$, any $n \geq 1$ and for any $N > 1$

$$\mathbb{E}\left(\left[\frac{\gamma_n^N(1)}{\gamma_n(1)} - 1\right]^2\right) \leq d \; \frac{n+1}{N-1} \; \left(1 + \frac{d}{N-1}\right)^{n-1} \tag{3.11}$$

The non asymptotic estimates stated in the above theorem extend the one presented in [3, 4] for Feynman-Kac type models (2.8) where $\mu_n = 0$. For such models, the $\mathbb{L}_r$-mean error estimates (3.9) are satisfied with the collection of parameters $b_{p,n} := 2q_{p,n}^2 \; \beta(P_{p,n})$, with $p \leq n$. The extra terms in (3.10) are intimately related to μ_n whose effects in the semigroup stability depend on the nature of G_n. We refer to theorem 3.1, section 4.2 and section 4.3, for a discussion on three different behaviors in the three cases presented in (3.4).

A direct consequence of this theorem is that it implies the almost sure convergence results:

$$\lim_{N\to\infty} \eta_n^N(f) = \eta_n(f) \quad \text{and} \quad \lim_{N\to\infty} \gamma_n^N(f) = \gamma_n(f)$$

for any bounded function $f \in \mathcal{B}(E_n)$.

Our last main result is a functional central limit theorem. We let W_n^N be the centered random fields defined by the following formula

$$\eta_n^N = \eta_{n-1}^N K_{n,(\gamma_n^N(1),\eta_{n-1}^N)} + \frac{1}{\sqrt{N}} \; W_n^N \; . \tag{3.12}$$

We also consider the pair of random fields

$$V_n^{\eta,N} := \sqrt{N}[\eta_n^N - \eta_n] \quad \text{and} \quad V_n^{\gamma,N} := \sqrt{N}[\gamma_n^N - \gamma_n]$$

For $n = 0$, we use the convention $W_0^N = V_0^{\eta,N}$.

Theorem 3.3 *The sequence of random fields $(W_n^N)_{n\geq 0}$ converges in law, as N tends to infinity, to the sequence of n independent, Gaussian and centered random fields $(W_n)_{n\geq 0}$ with a covariance function given for any $f, g \in \mathcal{B}(E_n)$ and $n \geq 0$ by*

$$\mathbb{E}(W_n(f)W_n(g))$$

$$= \eta_{n-1}K_{n,(\gamma_{n-1}(1),\eta_{n-1})}\left([f - K_{n,(\gamma_{n-1}(1),\eta_{n-1})}(f)][g - K_{n,(\gamma_{n-1}(1),\eta_{n-1})}(g)]\right). \tag{3.13}$$

In addition, the pair of random fields $V_n^{\gamma,N}$ and $V_n^{\eta,N}$ converge in law as $N \to \infty$ to a pair of centered Gaussian fields V_n^γ and V_n^η defined by

$$V_n^\gamma(f) := \sum_{p=0}^n \gamma_p(1) \, W_p(Q_{p,n}(f)) \quad and \quad V_n^\eta(f) := V_n^\gamma \left(\frac{1}{\gamma_n(1)}(f - \eta_n(f)) \right)$$

The details of the proof of theorem 3.2 and theorem 3.3 can be found in section 5.2. The proof of the non-asymptotic variance estimate (3.11) is given in section 5.2.1 dedicated to the convergence of the unnormalized particle measures γ_n^N. The $\mathbb{L}_r$-mean error estimates (3.9) and the fluctuation theorem 3.3 are proved in section 5.2.2. Under additional regularity conditions, we conjecture that it is possible to obtain uniform estimates for theorem 3.3 but have not established it here.

The rest of the article is organized as follows.

In section 4, we analyze the semigroup properties of the total mass process $\gamma_n(1)$ and the sequence of probability distributions η_n. This section is mainly concerned with the proof of theorem 3.1. The long time behavior of the total mass process is discussed in section 4.1, while the asymptotic behavior of the probability distributions is discussed in section 4.2. In section 4.3, we develop a series of Lipschitz type functional inequalities for uniform estimates w.r.t. the time parameter for the particle approximation. In section 5, we present the McKean models associated to the sequence $(\gamma_n(1), \eta_n)$ and their mean field particle interpretations. Section 5.2 is concerned with the convergence analysis of these particle approximations. In section 5.2.1, we discuss the convergence of the approximations of $\gamma_n(1)$, including their unbiasedness property and the non asymptotic variance estimates presented in (3.11). The proof of the $\mathbb{L}_r$-mean error estimates (3.9) is presented in section 5.2.2. The proof of the functional central limit theorem 3.3 is a more or less direct consequence of the decomposition formulae presented in section 5.2 and is just sketched at the end of this very section.

4   Semigroup analysis

The purpose of this section is to analyze the semigroup properties of the intensity measure recursion (2.2). We establish a framework for the analysis of the long time behavior of these measures and their particle approximations (2.9). First, we briefly recall some estimate of the quantities $(q_{p,n}, \beta(P_{p,n}))$ in terms of the potential functions G_n and the Markov transitions M_n. Further details on this subject can be found in [4], and in references therein.

We assume here that the following condition is satisfied for some $k \geq 1$, some collection of numbers $\epsilon_p \in (0,1)$

$$(M)_k \quad M_{p,p+k}(x_p, \cdot) \geq \epsilon_p \, M_{p,p+k}(y_p, \cdot) \quad \text{with} \quad M_{p,p+k} = M_{p+1}M_{p+2}\ldots M_{p+k} \tag{4.1}$$

for any time parameter p and any pair of states $(x_p, y_p) \in E_p^2$. It is well known that the mixing type condition $(M)_k$ is satisfied for any aperiodic and irreducible Markov chains on finite spaces, as well as for bi-Laplace exponential transitions associated with a bounded drift function and for Gaussian transitions with a mean drift function that is constant outside some compact domain. We introduce the following quantities

$$\delta_{p,n} := \sup \prod_{p \leq q < n} (G_q(x_q)/G_q(y_q)) \quad \text{and} \quad \delta_p^{(k)} := \delta_{p+1,p+k} \tag{4.2}$$

where the supremum is taken over all admissible pair of paths with transitions M_q where an admissible path $(x_{p-1}, x_{p+1}, ..., x_{n-1})$ is such that $\prod_{p \leq q < n} M_q(x_{q-1}, dx_q) > 0$. Under the

above conditions, we have [4, p. 140]

$$\beta(P_{p,p+n}) \le \prod_{l=0}^{\lfloor n/k \rfloor - 1} \left(1 - \epsilon_{p+lk}^2 / \delta_{p+lk}^{(k)}\right) \quad \text{and} \quad q_{p,p+n} \le \delta_{p,p+k}/\epsilon_p \tag{4.3}$$

For time-homogeneous Feynman-Kac models we set $\epsilon := \epsilon_k$ and $\delta_k := \delta_{0,k}$, for any $k \ge 0$. Using this notation, the above estimates reduce to [4, p. 142]

$$q_{p,p+n} \le \delta_k/\epsilon \quad \text{and} \quad \beta(P_{p,p+n}) \le \left(1 - \epsilon^2/\delta_{k-1}\right)^{\lfloor n/k \rfloor} \tag{4.4}$$

4.1 Description of the models

The next proposition gives a Markov transport formulation of Γ_n introduced in (2.5).

Proposition 4.1 *For any $n \ge 0$, we have the recursive formula*

$$\begin{cases} \gamma_{n+1}(1) & = \quad \gamma_n(1) \ \eta_n(G_n) + \mu_{n+1}(1) \\[2mm] \eta_{n+1} & = \quad \Psi_{G_n}(\eta_n) M_{n+1,(\gamma_n(1),\eta_n)} \end{cases} \tag{4.5}$$

with the collection of Markov transitions $M_{n+1,(m,\eta)}$ indexed by the parameters $m \in \mathbb{R}_+$ and the probability measures $\eta \in \mathcal{P}(E_n)$ given below

$$M_{n+1,(m,\eta)}(x, dy) := \alpha_n(m, \eta) M_{n+1}(x, dy) + (1 - \alpha_n(m, \eta)) \ \overline{\mu}_{n+1}(dy) \tag{4.6}$$

with the collection of $[0,1]$-parameters $\alpha_n(m, \eta)$ defined below

$$\alpha_n(m, \eta) = \frac{m\eta(G_n)}{m\eta(G_n) + \mu_{n+1}(1)}$$

Proof:
Observe that for any function $f \in \mathcal{B}(E_{n+1})$, we have that

$$\eta_{n+1}(f) = \frac{\gamma_n(G_n M_{n+1}(f)) + \mu_{n+1}(f)}{\gamma_n(G_n) + \mu_{n+1}(1)} = \frac{\gamma_n(1) \ \eta_n(G_n M_{n+1}(f)) + \mu_{n+1}(f)}{\gamma_n(1) \ \eta_n(G_n) + \mu_{n+1}(1)}$$

from which we find that

$$\eta_{n+1} = \alpha_n(\gamma_n(1), \eta_n) \ \Phi_{n+1}(\eta_n) + (1 - \alpha_n(\gamma_n(1), \eta_n)) \ \overline{\mu}_{n+1}$$

From these observations, we prove (4.5). This ends the proof of the proposition. ■

We let Γ_{n+1} be the mapping from $\mathbb{R}_+ \times \mathcal{P}(E_n)$ into $\mathbb{R}_+ \times \mathcal{P}(E_{n+1})$ given by

$$\Gamma_{n+1}(m, \eta) = \left(\Gamma_{n+1}^1(m, \eta), \Gamma_{n+1}^2(m, \eta)\right) \tag{4.7}$$

with the pair of transformations:

$$\Gamma_{n+1}^1(m, \eta) = m \ \eta(G_n) + \mu_{n+1}(1) \quad \text{and} \quad \Gamma_{n+1}^2(m, \eta) = \Psi_{G_n}(\eta) M_{n+1,(m,\eta)}$$

We also denote by $(\Gamma_{p,n})_{0 \le p \le n}$ the corresponding semigroup defined by

$$\forall 0 \le p \le n \qquad \Gamma_{p,n} = \Gamma_{p+1,n} \Gamma_{p+1} = \Gamma_n \Gamma_{n-1} \ldots \Gamma_{p+1}$$

with the convention $\Gamma_{n,n} = Id$.
  The following lemma collects some important properties of the sequence of intensity measures γ_n.

Lemma 4.2 *For any $0 \leq p \leq n$, we have the semigroup decomposition*

$$\gamma_n = \gamma_p Q_{p,n} + \sum_{p<q\leq n} \mu_q Q_{q,n} \quad and \quad \gamma_n = \sum_{0\leq p\leq n} \mu_p Q_{p,n} \tag{4.8}$$

In addition, we also have the following formula

$$\gamma_n(1) = \sum_{p=0}^{n} \mu_p(1) \prod_{p\leq q<n} \eta_q(G_q) \tag{4.9}$$

Proof:

The first pair of formulae are easily proved using a simple induction, and recalling that $\gamma_0 = \mu_0$. To prove the last assertion, we use an induction on the parameter $n \geq 0$. The result is obvious for $n = 0$. We also have by (2.2)

$$\gamma_{n+1}(1) = \gamma_n Q_{n+1}(1) + \mu_{n+1}(1) = \gamma_n(G_n) + \mu_{n+1}(1)$$

This implies

$$\begin{aligned}
\gamma_{n+1}(1) &= \gamma_n(1) \ \eta_n(G_n) + \mu_{n+1}(1) \\
&= \gamma_{n-1}(1) \ \eta_{n-1}(G_{n-1}) \ \eta_n(G_n) + \mu_n(1) \ \eta_n(G_n) + \mu_{n+1}(1) \\
&= \ldots \\
&= \gamma_0(1) \prod_{p=0}^{n} \eta_p(G_p) \ + \sum_{p=1}^{n+1} \mu_p(1) \prod_{p\leq q\leq n} \eta_q(G_q)
\end{aligned}$$

Recalling that $\gamma_0(dx_0) = \mu_0(dx_0)$, we prove (4.9). This ends the proof of the lemma. ∎

Using lemma 4.2, one proves that the semigroup $\Gamma_{p,n}$ satisfies the pair of formulae described below

Proposition 4.3 *For any $0 \leq p \leq n$, we have*

$$\Gamma_{p,n}^1(m, \eta) = m \ \eta Q_{p,n}(1) + \sum_{p<q\leq n} \mu_q Q_{q,n}(1) \tag{4.10}$$

$$\Gamma_{p,n}^2(m, \eta) = \alpha_{p,n}(m, \eta) \ \Phi_{p,n}(\eta) + (1 - \alpha_{p,n}(m, \eta)) \sum_{p<q\leq n} \frac{c_{q,n}}{\sum_{p<r\leq n} c_{r,n}} \ \Phi_{q,n}(\overline{\mu}_q) \tag{4.11}$$

with the collection of parameters $c_{p,n} := \mu_p Q_{p,n}(1)$ and the $[0,1]$-valued parameters $\alpha_{p,n}(m, \eta)$ defined below

$$\alpha_{p,n}(m, \eta) = \frac{m\eta Q_{p,n}(1)}{m\eta Q_{p,n}(1) + \sum_{p<q\leq n} c_{q,n}} \leq \alpha_{p,n}^\star(m) := 1 \wedge \left[m \left\| \frac{Q_{p,n}(1)}{\sum_{p<q\leq n} c_{q,n}} \right\| \right] \tag{4.12}$$

One central question in the theory of spatial branching point processes is the long time behavior of the total mass process $\gamma_n(1)$. Notice that $\gamma_n(1) = \mathbb{E}(\mathcal{X}_n(1))$ is the expected size of the n-th generation. For time homogeneous models with null spontaneous branching $\mu_n = \mu = 0$, the exponential growth of these quantities are related to the logarithmic Lyapunov exponents of the semigroup $Q_{p,n}$. The prototype of these models is the Galton-Watson

branching process. In this context three typical situations may occur: 1) $\gamma_n(1)$ remains constant and equals to the initial mean number of individuals. 2) $\gamma_n(1)$ goes exponentially fast to 0, 3) $\gamma_n(1)$ grows exponentially fast to infinity,

The analysis of spatial branching point processes with $\mu_n = \mu \neq 0$ considered here is more involved. Loosely speaking, in the first situation discussed above the total mass process is generally strictly increasing; while in the second situation the additional mass injected in the system stabilizes the total mass process. Before giving further details, by lemma 4.2 we observe $\gamma_n(1) \in I_n$, for any $n \geq 0$, with the compact interval I_n defined in 3.8.

We end this section with a more precise analysis of the effect of μ in the three scenarios (3.4).

In the further developments of this section, we illustrate the stability properties of the sequence of probability distributions η_n in these three scenarios.

1. When $G(x) = 1$ for any $x \in E$, the total mass process $\gamma_n(1)$ grows linearly w.r.t. the time parameter and we have

$$\gamma_n(1) = m_-(n) = m_+(n) = \gamma_0(1) + \mu(1)\ n \tag{4.13}$$

Note that the estimates in (4.12) take the following form

$$\alpha_{p,n}\left(\gamma_p(1), \eta_p\right) \leq \alpha_{p,n}^\star(\gamma_p(1)) := 1 \wedge \frac{\gamma_0(1) + \mu(1)\ p}{\mu(1)\ (n-p)} \to_{(n-p)\to\infty} 0$$

2. When $g_+ < 1$, the total mass process $\gamma_n(1)$ is uniformly bounded w.r.t. the time parameter. More precisely, we have that

$$m_{-/+}(n) = g_{-/+}^n\ \gamma_0(1) + \left(1 - g_{-/+}^n\right)\ \frac{\mu(1)}{1 - g_{-/+}}$$

This yields the rather crude estimates

$$\gamma_0(1) \wedge \frac{\mu(1)}{1 - g_-} \leq \gamma_n(1) \leq \gamma_0(1) \vee \frac{\mu(1)}{1 - g_+} \tag{4.14}$$

We end this discussion with an estimate of the parameter $\alpha_{p,n}(m)$ given in (4.12). When the mixing condition $(M)_k$ stated in (4.1) is satisfied for some k and some fixed parameters $\epsilon_p = \epsilon$, using (4.4) we prove that

$$\sum_{p<r\leq n} \frac{\mu Q_{r,n}(1)}{Q_{p,r}(Q_{r,n}(1))} \geq \frac{\epsilon\mu(1)}{\delta_k}\ \sum_{p<r\leq n} \frac{1}{Q_{p,r}(1)} \geq \frac{\epsilon\mu(1)}{\delta_k}\ \frac{g_+^{-(n-p)} - 1}{1 - g_+}$$

from which we conclude that for any $n > p$ and any $m \in I_p$

$$\alpha_{p,n}^\star(m) \leq 1 \wedge \left[m\ g_+^{(n-p)}\ \frac{\delta_k\ (1 - g_+)}{\epsilon\mu(1)(1 - g_+^{(n-p)})}\right]$$

$$\leq 1 \wedge \left[m\ g_+^{(n-p)}\ \delta_k/(\epsilon\mu(1))\right]$$

$$\leq 1 \wedge \left[\left(\gamma_0(1) \vee \frac{\mu(1)}{1 - g_+}\right)\ g_+^{(n-p)}\ \delta_k/(\epsilon\mu(1))\right] \to_{(n-p)\to\infty} 0 \tag{4.15}$$

3. When $g_- > 1$, the total mass process $\gamma_n(1)$ grows exponentially fast w.r.t. the time parameter and we can easily show that

$$g_- > 1 \implies \gamma_n(1) \geq m_-(n) = \gamma_0(1)\ g_-^n + \mu(1)\ \frac{g_-^n - 1}{g_- - 1} \tag{4.16}$$

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4.2 Asymptotic properties

This section is concerned with the long time behavior of the semigroups $\Gamma_{p,n}$ in the three scenarios discussed in (4.13), (4.14), and (4.16). Our results are summarized in theorem 3.1. We consider time-homogeneous models $(E_n, G_n, M_n, \mu_n) = (E, G, M, \mu)$.

1. When $G(x) = 1$ for any $x \in E$, we have seen in (4.13) that $\gamma_n(1) = \gamma_0(1) + \mu(1)\ n$. In this particular situation, the time-inhomogeneous Markov transitions $M_{n,(\gamma_{n-1}(1),\eta_{n-1})} := \overline{M}_n$ introduced in (4.5) are given by

$$\overline{M}_n(x, dy) = \left(1 - \frac{\mu(1)}{\gamma_0(1) + n\mu(1)}\right) M(x, dy) + \frac{\mu(1)}{\gamma_0(1) + n\mu(1)}\ \overline{\mu}(dy)$$

This shows that $\eta_n = \text{Law}(\overline{X}_n)$ can be interpreted as the distribution of the states $\overline{X}_n$ of a time inhomogeneous Markov chain with transitions $\overline{M}_n$ and initial distribution η_0. If we choose in (2.6) $K_{n+1,(\gamma_n(1),\eta_n)} = \overline{M}_{n+1}$, the N-particle model (2.9) reduces to a series of N independent copies of $\overline{X}_n$. In this situation, the mapping $\Gamma^2_{0,n}$ is given by

$$\Gamma^2_{0,n}(\gamma_0(1), \eta_0) := \frac{\gamma_0(1)}{\gamma_0(1) + n\mu(1)}\ \eta_0 M^n + \frac{n\mu(1)}{\gamma_0(1) + n\mu(1)}\ \frac{1}{n}\sum_{0 \le p < n} \overline{\mu}M^p$$

The above formula shows that for a large time horizon n, the normalized distribution flow η_n is almost equal to $\frac{1}{n}\sum_{0 \le p < n} \overline{\mu}M^p$. Let us assume that the Markov kernel M is chosen so that (3.6) is satisfied for some invariant measure $\eta_\infty = \eta_\infty M$. In this case, for any starting measure γ_0, we have

$$\|\eta_n - \eta_\infty\|_{\text{tv}} \le \frac{\gamma_0(1)}{\gamma_0(1) + n\mu(1)}\ \tau_n + \frac{n\mu(1)}{\gamma_0(1) + n\mu(1)}\ \frac{1}{n}\sum_{0 \le p < n} \tau_p = O\left(\frac{1}{n}\right)$$

with $\tau_n = \sup_{x \in E} \|M^n(x, .) - \eta_\infty\|_{\text{tv}}$. For instance, suppose the mixing condition $(M)_k$ presented in (4.1) is met for some $k \ge 1$ and $\epsilon > 0$. In this case, the above upper bound is satisfied with $\tau_n = (1 - \epsilon)^{\lfloor n/k \rfloor}$.

2. Consider the case where $g_+ < 1$. In this situation, the pair of measures (3.7) are well defined. Furthermore, for any $f \in \mathcal{B}(E)$ with $\|f\| \le 1$, we have the estimates

$$|\gamma_n(f) - \gamma_\infty(f)| \le \gamma_0(1)\ \eta_0 Q^n(1) + \sum_{p \ge n} \mu Q^p(1)$$

$$\le g_+^n\ [\gamma_0(1) + \mu(1)/(1 - g_+)] \longrightarrow_{n \to \infty} 0$$

In addition, using the fact that $\gamma_n(1) \ge \mu(1)$, we find that for any $f \in \text{Osc}_1(E)$

$$|\eta_n(f) - \eta_\infty(f)| \le \frac{1}{\gamma_n(1)}\ |\gamma_n[f - \eta_\infty(f)] - \gamma_\infty[f - \eta_\infty(f)]|$$

$$\le g_+^n\ [\gamma_0(1)/\mu(1) + 1/(1 - g_+)] \longrightarrow_{n \to \infty} 0$$

3. Consider the case where $g_- > 1$. We further assume that the mixing condition $(M)_k$ presented in (4.1) is met for some $k \ge 1$ and some fixed parameters $\epsilon_p = \epsilon > 0$. In this situation, it is well known that the mapping $\Phi = \Phi_{n-1,n}$ introduced in (3.2) has a unique fixed point $\eta_\infty = \Phi(\eta_\infty)$, and for any initial distribution η_0, we have

$$\|\Phi_{0,n}(\eta_0) - \eta_\infty\|_{\text{tv}} \le a\ e^{-\lambda\ n} \tag{4.17}$$

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with

$$\lambda = -\frac{1}{k} \log\left(1 - \epsilon^2/\delta_{0,k-1}\right) \quad \text{and} \quad a = 1/\left(1 - \epsilon^2/\delta_{0,k-1}\right)$$

as well as

$$\sup_{\eta \in \mathcal{P}(E)} \left| \frac{1}{n} \log \eta Q^n(1) - \log \eta_\infty(G) \right| \leq b/n \tag{4.18}$$

for some finite constant $b < \infty$. For a more thorough discussion on the stability properties of the semigroup $\Phi_{0,n}$ and the limiting measures η_∞, we refer the reader to [4]. Our next objective is to transfer these stability properties to the one of the sequence η_n. First, using (4.18), we readily prove that

$$\lim_{n\to\infty} \frac{1}{n} \log \gamma_n(1) = \log \eta_\infty(G)$$

Next, we simplify the notation and we set $\alpha_n := \alpha_{0,n}(\gamma_0(1), \eta_0)$ and $c_n := c_{0,n}$. Using (4.11), we find that for any $n > 1$

$$a^{-1} \|\eta_n - \eta_\infty\|_{\mathrm{tv}} \leq \alpha_n\, e^{-\lambda n} + (1 - \alpha_n) \sum_{0 \leq p < n} \frac{c_p}{\sum_{0 \leq q < n} c_q}\, e^{-\lambda p}$$

Recalling that

$$\mu(1)\, g_-^p \leq c_p = \mu Q^p(1) \leq \mu(1)\, g_+^p$$

we also obtain that

$$\sum_{0 \leq p < n} \frac{c_p}{\sum_{1 \leq q < n} c_q}\, e^{-\lambda p} \leq \frac{1}{\left[\sum_{0 \leq q < n} c_q\right]^{1/r}} \left[\sum_{0 \leq p < n} c_p e^{-\lambda p r}\right]^{1/r}$$

$$\leq \frac{1}{\left[\sum_{0 \leq q < n} g_-^q\right]^{1/r}} \left[\sum_{0 \leq p < n} (e^{-\lambda r} g_+)^p\right]^{1/r} \tag{4.19}$$

for any $r \geq 1$. We conclude that

$$r > \frac{1}{\lambda}\, \log g_+ \implies \sum_{0 \leq p < n} \frac{c_p}{\sum_{0 \leq q < n} c_q}\, e^{-\lambda p} \leq g_-^{-(n-1)/r}/(1 - e^{-\lambda r} g_+)^{1/r}$$

and therefore

$$a^{-1} \|\eta_n - \eta_\infty\|_{\mathrm{tv}} \leq e^{-\lambda n} + g_-^{-(n-1)/r}/(1 - e^{-\lambda r} g_+)^{1/r} \to_{n\to\infty} 0$$

4.3   Stability and Lipschitz regularity properties

We describe in this section a framework that allows to transfer the regularity properties of the Feynman-Kac semigroups $\Phi_{p,n}$ introduced in (3.2) to the ones of the semigroup $\Gamma_{p,n}$ of the sequence $(\gamma_n(1), \eta_n)$. Before proceeding we recall a lemma that provides some weak Lipschitz type inequalities for the Feynman-Kac semigroup $\Phi_{p,n}$ in terms of the Dobrushin contraction coefficient associated with the Markov transitions $P_{p,n}$ introduced in (3.2). The details of the proof of this result can be found in [4] or in [5] (see Lemma 4.4. in [5], or proposition 4.3.7 on page 146 in [4]).

Lemma 4.4 ([5]) *For any $0 \le p \le n$, any $\eta, \mu \in \mathcal{P}(E_p)$ and any $f \in \mathrm{Osc}_1(E_n)$, we have*

$$\left| [\Phi_{p,n}(\mu) - \Phi_{p,n}(\eta)] (f) \right| \le 2 \, q_{p,n}^2 \, \beta(P_{p,n}) \, |(\mu - \eta) \mathcal{D}_{p,n,\eta}(f)| \tag{4.20}$$

for a collection of functions $\mathcal{D}_{p,n,\eta}(f) \in \mathrm{Osc}_1(E_p)$ whose values only depend on the parameters (p, n, η).

Proposition 4.5 *For any $0 \le p \le n$, any $\eta, \eta' \in \mathcal{P}(E_p)$ and any $f \in \mathrm{Osc}_1(E_n)$, there exits a collection of functions $\mathcal{D}_{p,n,\eta'}(f) \in \mathrm{Osc}_1(E_p)$ whose values only depend on the parameters (p, n, η) and such that, for any $m \in I_p$, we have*

$$\left| \left[\Gamma_{p,n}^2(m, \eta) - \Gamma_{p,n}^2(m, \eta') \right] (f) \right|$$
$$\le 2 \, \alpha_{p,n}^\star \, q_{p,n} \left[q_{p,n} \, \beta(P_{p,n}) \, \left| (\eta - \eta') \mathcal{D}_{p,n,\eta'}(f) \right| + \beta_{p,n} \left| (\eta - \eta') h_{p,n,\eta'} \right| \right] \tag{4.21}$$

with the collection of functions $h_{p,n,\eta'} = \frac{1}{2q_{p,n}} \frac{Q_{p,n}(1)}{\eta' Q_{p,n}(1)} \in \mathrm{Osc}_1(E_p)$ and the sequence of parameters $\epsilon_{p,n}$ and $\beta_{p,n}$ defined below

$$\alpha_{p,n}^\star := \alpha_{p,n}^\star(m_+(p)) \quad and \quad \beta_{p,n} := \sum_{p < q \le n} \frac{c_{q,n}}{\sum_{p < r \le n} c_{r,n}} \, \beta(P_{q,n}) \tag{4.22}$$

Before getting into the details of the proof of proposition 4.5, we illustrate some consequences of these weak functional inequalities for time-homogeneous models $(E_n, G_n, M_n, \mu_n) = (E, G, M, \mu)$ in the three scenarios discussed in (4.13), (4.14), and (4.16).

1. When $G(x) = 1$ for any $x \in E$, we have

$$\Phi_{p,n}(\eta) = \eta M^{(n-p)}, \quad h_{p,n,\eta'} = 1/2 \quad c_{p,n} = \mu(1) \quad q_{p,n} = 1 \quad \alpha_{p,n}^\star \le 1$$

Let us assume that there exist $a < \infty$ and $0 < \lambda < \infty$ such that $\beta(M^n) \le a e^{-\lambda n}$ for any $n \ge 0$. In this situation, we prove using (4.21) that

$$\left| \left[\Gamma_{p,n}^2(m, \eta) - \Gamma_{p,n}^2(m, \eta') \right] (f) \right| \le 2 a e^{-\lambda(n-p)} \, |(\mu - \eta) \mathcal{D}_{p,n,\eta'}(f)|$$

2. When $g_+ < 1$ and when the mixing condition $(M)_k$ stated in (4.1) is satisfied for some k and some fixed parameters $\epsilon_p = \epsilon$, we have seen in (4.15) that

$$\sup_{m \in I_p} \alpha_{p,n}^\star(m) \le 1 \wedge \left(d \, g_+^{(n-p)} \right) \quad with \quad d = \left((\gamma_0(1)/\mu(1)) \vee (1 - g_+)^{-1} \right) \delta_{0,k} \epsilon^{-1}$$

Furthermore, using the estimates given in (4.3) and (4.4), we also have that

$$q_{p,n} \le \delta_k/\epsilon \qquad \beta_{p,n} \le 1 \quad and \quad \beta(P_{p,n}) \le a \, e^{-\lambda \, (n-p)} \quad with \ (a, \lambda) \ given \ in \ (4.17)$$

In this situation, we prove using (4.21) that

$$\left| \left[\Gamma_{p,n}^2(m, \eta) - \Gamma_{p,n}^2(m, \eta') \right] (f) \right|$$

$$\le 2 \left[1 \wedge \left(d \, g_+^{(n-p)} \right) \right] (\delta_k/\epsilon) \left[(\delta_k/\epsilon) \, a \, e^{-\lambda(n-p)} \, |(\mu - \eta) \mathcal{D}_{p,n,\eta'}(f)| + |(\mu - \eta) h_{p,n,\eta'}| \right]$$

Notice that for $(n - p) \ge \log(d)/\log(1/g_+)$, this yields

$$\left| \left[\Gamma_{p,n}^2(m, \eta) - \Gamma_{p,n}^2(m, \eta') \right] (f) \right|$$

$$\le a_0 \, e^{-\lambda_0(n-p)} \, |(\mu - \eta) \mathcal{D}_{p,n,\eta'}(f)| + a_1 \, e^{-\lambda_1(n-p)} \, |(\mu - \eta) h_{p,n,\eta'}|$$

with

$$a_0 = 2ad(\delta_k/\epsilon)^2 \quad a_1 = 2d(\delta_k/\epsilon) \quad \lambda_0 = \lambda + \log(1/g_+) \quad and \quad \lambda_1 = \log(1/g_+)$$

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3. When $g_- > 1$ and when the mixing condition $(M)_k$ presented in (4.1) is met for some k and some fixed parameters $\epsilon_p = \epsilon > 0$, then we use the fact that

$$\alpha^\star_{p,n} \leq 1 \quad q_{p,n} \leq \delta_k/\epsilon \quad \text{and} \quad \beta(P_{p,n}) \leq a \, e^{-\lambda(n-p)} \quad \text{with } (a,\lambda) \text{ given in (4.17)}$$

Arguing as in (4.19), we prove that for any $r > \frac{1}{\lambda} \log g_+$

$$\beta_{p,n} \leq g_-^{-(n-p-1)/r}/(1 - e^{-\lambda r} g_+)^{1/r}$$

from which we conclude that

$$\left| \left[\Gamma^2_{p,n}(m,\eta) - \Gamma^2_{p,n}(m,\eta') \right](f) \right|$$

$$\leq a_0 \, e^{-\lambda_0(n-p)} \left| (\mu - \eta)\mathcal{D}_{p,n,\eta'}(f) \right| + a_1 \, e^{-\lambda_1(n-p)} \left| (\mu - \eta)h_{p,n,\eta'} \right|$$

with

$$a_0 = 2a(\delta_k/\epsilon)^2 \quad a_1 = 2g_-^r(\delta_k/\epsilon)/(1 - e^{-\lambda r} g_+)^{1/r} \quad \lambda_0 = \lambda \quad \text{and} \quad \lambda_1 = \log(g_-)$$

Now, we come to the proof of proposition 4.5.

Proof of proposition 4.5:

First, we observe that

$$\Gamma^2_{p,n}(m,\eta) - \Gamma^2_{p,n}(m',\eta')$$

$$= \alpha_{p,n}(m,\eta) \left[\Phi_{p,n}(\eta) - \sum_{p<q\leq n} \frac{c_{q,n}}{\sum_{p<r\leq n} c_{r,n}} \Phi_{q,n}(\overline{\mu}_q) \right]$$

$$- \alpha_{p,n}(m',\eta') \left[\Phi_{p,n}(\eta') - \sum_{p<q\leq n} \frac{c_{q,n}}{\sum_{p<r\leq n} c_{r,n}} \Phi_{q,n}(\overline{\mu}_q) \right]$$

Using the following decomposition

$$ab - a'b' = a'(b - b') + (a - a')b' + (a - a')(b - b') \tag{4.23}$$

which is valid for any $a, a', b, b' \in \mathbb{R}$, we prove that

$$\Gamma^2_{p,n}(m,\eta) - \Gamma^2_{p,n}(m',\eta')$$

$$= \alpha_{p,n}(m',\eta') \left[\Phi_{p,n}(\eta) - \Phi_{p,n}(\eta') \right]$$

$$\tag{4.24}$$

$$+ \left[\Phi_{p,n}(\eta') - \sum_{p<q\leq n} \frac{c_{q,n}}{\sum_{p<r\leq n} c_{r,n}} \Phi_{q,n}(\overline{\mu}_q) \right] \left[\alpha_{p,n}(m,\eta) - \alpha_{p,n}(m',\eta') \right]$$

$$+ \left[\alpha_{p,n}(m,\eta) - \alpha_{p,n}(m',\eta') \right] \left[\Phi_{p,n}(\eta) - \Phi_{p,n}(\eta') \right]$$

For $m = m'$, using (4.24) we find that

$$\Gamma^2_{p,n}(m,\eta) - \Gamma^2_{p,n}(m,\eta')$$

$$= \alpha_{p,n}(m,\eta) \left[\Phi_{p,n}(\eta) - \Phi_{p,n}(\eta') \right]$$

$$+ \left[\Phi_{p,n}(\eta') - \sum_{p<q\leq n} \frac{c_{q,n}}{\sum_{p<r\leq n} c_{r,n}} \Phi_{q,n}(\overline{\mu}_q) \right] \left[\alpha_{p,n}(m,\eta) - \alpha_{p,n}(m,\eta') \right]$$

We also notice that

$$\alpha_{p,n}(m,\eta) = \frac{1}{1 + \mu_{p,n}/[m\eta Q_{p,n}(1)]}$$

from which we easily prove that

$$\alpha_{p,n}(m, \eta) - \alpha_{p,n}(m', \eta')$$

$$= \frac{\mu_{p,n}}{\mu_{p,n} + m\eta Q_{p,n}(1)} \frac{1}{\mu_{p,n} + m'\eta' Q_{p,n}(1)} [m\eta Q_{p,n}(1) - m'\eta' Q_{p,n}(1)]$$

and therefore

$$\alpha_{p,n}(m, \eta) - \alpha_{p,n}(m, \eta') = (\alpha_{p,n}(m, \eta')(1 - \alpha_{p,n}(m, \eta)))\ [\eta - \eta']\left(\frac{Q_{p,n}(1)}{\eta' Q_{p,n}(1)}\right)$$

The proof of $\alpha_{p,n}(m, \eta) \leq \alpha^{\star}_{p,n}(m)$ is elementary. From the above decomposition, we prove the following upper bounds

$$\left|\alpha_{p,n}(m, \eta) - \alpha_{p,n}(m, \eta')\right| \leq \alpha^{\star}_{p,n}(m)\left|[\eta - \eta']\left(\frac{Q_{p,n}(1)}{\eta' Q_{p,n}(1)}\right)\right|$$

and

$$\left|\left[\Gamma^2_{p,n}(m, \eta) - \Gamma^2_{p,n}(m, \eta')\right](f)\right|$$

$$\leq \alpha^{\star}_{p,n}(m)\left[\left|[\Phi_{p,n}(\eta) - \Phi_{p,n}(\eta')](f)\right|\right.$$

$$\left. + \left|[\eta - \eta']\left(\frac{Q_{p,n}(1)}{\eta' Q_{p,n}(1)}\right)\right|\ \left|\sum_{p<q\leq n}\frac{c_{q,n}}{\sum_{p<r\leq n}c_{r,n}}\ \left[\Phi_{q,n}(\overline{\mu}_q) - \Phi_{q,n}(\Phi_{p,q}(\eta'))\right](f)\right|\right]$$

This yields

$$\left|\left[\Gamma^2_{p,n}(m, \eta) - \Gamma^2_{p,n}(m, \eta')\right](f)\right|$$

$$\leq \alpha^{\star}_{p,n}(m)\left[\left|[\Phi_{p,n}(\eta) - \Phi_{p,n}(\eta')](f)\right| + \beta_{p,n}\ \left|[\eta - \eta']\left(\frac{Q_{p,n}(1)}{\eta' Q_{p,n}(1)}\right)\right|\right]$$

The last formula comes from the fact that

$$\beta(P_{q,n}) := \sup_{\nu, \nu' \in \mathcal{P}(E_q)} \left\|\Phi_{q,n}(\nu) - \Phi_{q,n}(\nu')\right\|_{\text{tv}}$$

The proof of this result can be found in [4] (proposition 4.3.1 on page 134). The end of the proof is now a direct consequence of lemma 4.4. This ends the proof of the proposition. ∎

5 Mean field particle approximations

5.1 McKean particle interpretations

In proposition 4.1, the evolution equation (4.5) of the sequence of probability measures $\eta_n \rightsquigarrow \eta_{n+1}$ is a combination of an updating type transition $\eta_n \rightsquigarrow \Psi_{G_n}(\eta_n)$ and an integral transformation w.r.t. a Markov transition $M_{n+1,(\gamma_n(1),\eta_n)}$ that depends on the current total mass $\gamma_n(1)$ and the current probability distribution η_n. The operator $M_{n+1,(\gamma_n(1),\eta_n)}$ defined in (4.6) is a mixture of the Markov transition M_{n+1} and the spontaneous birth normalized measure $\overline{\mu}_{n+1}$. We let S_{n,η_n} be any Markov transition from E_n into itself satisfying

$$\Psi_{G_n}(\eta_n) = \eta_n S_{n,\eta_n}$$

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The choice of these transitions is not unique. We can choose for instance one of the collection of transitions presented in (2.10), (2.11) and (2.12). Further examples of McKean acceptance-rejection type transitions can also be found in section 2.5.3 in [4]. By construction, we have the recursive formula

$$\eta_{n+1} = \eta_n K_{n+1,(\gamma_n(1),\eta_n)} \quad \text{with} \quad K_{n+1,(\gamma_n(1),\eta_n)} = S_{n,\eta_n} M_{n+1,(\gamma_n(1),\eta_n)} \tag{5.1}$$

with the auxiliary total mass evolution equation

$$\gamma_{n+1}(1) = \gamma_n(1) \ \eta_n(G_n) + \mu_{n+1}(1) \tag{5.2}$$

As already mentioned in section 2, the sequence of probability distributions η_n can be interpreted as the distributions of the states $\overline{X}_n$ of a process defined, conditional upon $(\gamma_n(1),\eta_n)$, by the elementary transitions

$$\mathbb{P}\left(\overline{X}_{n+1} \in dx \mid \overline{X}_n\right) = K_{n,(\gamma_n(1),\eta_n)}\left(\overline{X}_n, dx\right) \quad \text{with} \quad \eta_n = \text{Law}(\overline{X}_n)$$

Next, we define the mean field particle interpretations of the sequence $(\gamma_n(1),\eta_n)$ given in (5.1) and (5.2). First, mimicking formula (5.2) we set

$$\gamma_{n+1}^N(1) := \gamma_n^N(1) \ \eta_n^N(G_n) + \mu_{n+1}(1) \quad \text{and} \quad \gamma_n^N(f) = \gamma_n^N(1) \ \times \ \eta_n^N(f)$$

for any $f \in \mathcal{B}(E_n)$, with the initial measure $\gamma_0^N = \gamma_0$. It is important to notice that

$$\gamma_n^N(1) = \gamma_0(1) \prod_{0 \le q < n} \eta_q^N(G_q) + \sum_{p=1}^n \mu_p(1) \prod_{p \le q < n} \eta_q^N(G_q) \Longrightarrow \gamma_n^N(1) \in I_n$$

The mean field particle interpretation of the nonlinear measure valued model (5.1) is an E_n^N-valued process ξ_n with elementary transitions defined in (2.9) and (5.1). By construction, the particle evolution is a simple combination of a selection and a mutation genetic type transition

$$\xi_n \rightsquigarrow \widehat{\xi}_n = (\widehat{\xi}_n^i)_{1 \le i \le N} \rightsquigarrow \xi_{n+1}$$

During the selection transitions $\xi_n \rightsquigarrow \widehat{\xi}_n$, each particle $\xi_n^i \rightsquigarrow \widehat{\xi}_n^i$ evolves according to the selection type transition $S_{n,\eta_n^N}(\xi_n^i, dx)$. During the mutation stage, each of the selected particles $\widehat{\xi}_n^i \rightsquigarrow \xi_{n+1}^i$ evolves according to the transition

$$M_{n+1,(\gamma_n^N(1),\eta_n^N)}(x,dy) := \alpha_n\left(\gamma_n^N(1),\eta_n^N\right) M_{n+1}(x,dy) + \left(1 - \alpha_n\left(\gamma_n^N(1),\eta_n^N\right)\right) \overline{\mu}_{n+1}(dy)$$

5.2 Asymptotic behavior

This section is mainly concerned with the proof of theorem 3.2. In section 5.2.1, we discuss the unibiasedness property of the particle measures γ_n^N and their convergence properties towards γ_n, as the number of particles N tends to infinity. We mention that the proof of the non asymptotic variance estimates (3.11) is simpler than the one provided in a recent article by the second author with F. Cérou and A. Guyader [3]. Section 5.2.2 is concerned with the convergence and the fluctuations of the occupation measures η_n^N around their limiting measures η_n.

5.2.1 Intensity measures

We start this section with a simple unbiasedness property. Recall that $\mathcal{F}_p^{(N)}$ stands for the σ-field generated by the random sequence $(\xi_k^{(N)})_{0 \leq k \leq p}$.

Proposition 5.1 *For any $0 \leq p \leq n$, and any $f \in \mathcal{B}(E_n)$, we have*

$$\mathbb{E}\left(\gamma_{n+1}^N(f) \,\middle|\, \mathcal{F}_p^{(N)}\right) = \gamma_p^N Q_{p,n+1}(f) + \sum_{p < q \leq n+1} \mu_q Q_{q,n+1}(f) \tag{5.3}$$

In particular, we have the unbiasedness property: $\mathbb{E}\left(\gamma_n^N(f)\right) = \gamma_n(f)$.

Proof:
By construction of the particle model, for any $f \in \mathcal{B}(E_n)$ we have

$$\mathbb{E}\left(\eta_{n+1}^N(f) \,\middle|\, \mathcal{F}_n^{(N)}\right) = \eta_n^N K_{n+1,(\gamma_n^N(1),\eta_n^N)}(f) = \Gamma_{n+1}^2\left(\gamma_n^N(1),\eta_n^N\right)(f)$$

with the second component Γ_{n+1}^2 of the transformation Γ_{n+1} introduced in 4.7. Using the fact that

$$\Gamma_{n+1}^2\left(\gamma_n^N(1),\eta_n^N\right)(f) = \frac{\gamma_n^N(1)\,\eta_n^N(Q_{n+1}(f)) + \mu_{n+1}(f)}{\gamma_n^N(1)\,\eta_n^N(Q_{n+1}(1)) + \mu_{n+1}(1)} = \frac{\gamma_n^N(Q_{n+1}(f)) + \mu_{n+1}(f)}{\gamma_n^N(Q_{n+1}(1)) + \mu_{n+1}(1)}$$

and

$$\gamma_{n+1}^N(1) = \gamma_n^N(1)\,\eta_n^N(G_n) + \mu_{n+1}(1) = \gamma_n^N(Q_{n+1}(1)) + \mu_{n+1}(1)$$

we prove that

$$\mathbb{E}\left(\gamma_{n+1}^N(f) \,\middle|\, \mathcal{F}_n^{(N)}\right) = \mathbb{E}\left(\gamma_{n+1}^N(1)\,\eta_{n+1}^N(f) \,\middle|\, \mathcal{F}_n^{(N)}\right) = \gamma_{n+1}^N(1)\,\mathbb{E}\left(\eta_{n+1}^N(f) \,\middle|\, \mathcal{F}_n^{(N)}\right)$$
$$= \gamma_n^N(Q_{n+1}(f)) + \mu_{n+1}(f)$$

This also implies that

$$\mathbb{E}\left(\gamma_{n+1}^N(f) \,\middle|\, \mathcal{F}_{n-1}^{(N)}\right) = \mathbb{E}\left(\gamma_n^N(Q_{n+1}(f)) \,\middle|\, \mathcal{F}_{n-1}^{(N)}\right) + \mu_{n+1}(f)$$
$$= \gamma_{n-1}^N(Q_n Q_{n+1}(f)) + \mu_n(Q_{n+1}(f)) + \mu_{n+1}(f)$$

Iterating the argument one proves (5.3). The end of the proof is now clear. ∎

The next theorem provides a key martingale decomposition and a rather crude non asymptotic variance estimate.

Theorem 5.2 *For any $n \geq 0$ and any function $f \in \mathcal{B}(E_n)$, we have the decomposition*

$$\sqrt{N}\,\left[\gamma_n^N - \gamma_n\right](f) = \sum_{p=0}^n \gamma_p^N(1)\,W_p^N(Q_{p,n}(f)) \tag{5.4}$$

In addition, if the mixing condition $(M)_k$ presented in (4.1) is met for some $k \geq 1$ and some constant parameters $\epsilon_p = \epsilon > 0$, then we have for any $N > 1$ and any $n \geq 1$

$$\mathbb{E}\left(\left[\frac{\gamma_n^N(1)}{\gamma_n(1)} - 1\right]^2\right) \leq \frac{n+1}{N-1}\,\frac{\delta_k^2}{\epsilon^2}\left(1 + \frac{\delta_k^2}{\epsilon^2(N-1)}\right)^{n-1} \tag{5.5}$$

Before presenting the proof of this theorem, we would like to make a couple of comments. On the one hand, we observe that the unbiasedness property follows directly from the decomposition (5.4). On the other hand, using Kintchine's inequality, for any $r \geq 1$, $p \geq 1$, and any $f \in \mathrm{Osc}_1(E_n)$ we have the almost sure estimates

$$\sqrt{N} \; \mathbb{E}\left(\left| W_p^N(f) \right|^r \; \Big| \mathcal{F}_{p-1}^{(N)} \right)^{\frac{1}{r}} \leq a_r$$

A detailed proof of these estimates can be found in [4], see also lemma 7.2 in [1] for a simpler proof by induction on the parameter N. From this elementary observation, and recalling that $\gamma_n^N(1) \in I_n$ for any $n \geq 0$, we find that

$$\sqrt{N} \; \mathbb{E}\left(\left| \left[\gamma_n^N - \gamma_n \right] (f) \right|^r \right)^{\frac{1}{r}} \leq a_r \; b_n$$

for some finite constant b_n whose values only depend on the time parameter n.

Now, we present the proof of theorem 5.2.

Proof of theorem 5.2:

We use the decomposition:

$$\gamma_{n+1}^N(f) - \gamma_{n+1}(f) = \left[\gamma_{n+1}^N(f) - \mathbb{E}\left(\gamma_{n+1}^N(f) \Big| \mathcal{F}_n^{(N)} \right) \right] + \left[\mathbb{E}\left(\gamma_{n+1}^N(f) \Big| \mathcal{F}_n^{(N)} \right) - \gamma_{n+1}(f) \right]$$

By (5.3), we find that

$$\gamma_{n+1}^N(f) - \mathbb{E}\left(\gamma_{n+1}^N(f) \Big| \mathcal{F}_n^{(N)} \right) = \gamma_{n+1}^N(f) - \left[\gamma_n^N(Q_{n+1}(f)) + \mu_{n+1}(f) \right]$$

Since we have

$$\gamma_n^N(Q_{n+1}(1)) + \mu_{n+1}(1) = \gamma_n^N(G_n) + \mu_{n+1}(1)$$
$$= \gamma_n^N(1) \; \eta_n^N(G_n) + \mu_{n+1}(1) = \gamma_{n+1}^N(1)$$

this implies that

$$\gamma_{n+1}^N(f) - \left[\gamma_n^N(Q_{n+1}(f)) + \mu_{n+1}(f) \right] = \gamma_{n+1}^N(1) \; \left[\eta_{n+1}^N(f) - \frac{\left[\gamma_n^N(Q_{n+1}(f)) + \mu_{n+1}(f) \right]}{\left[\gamma_n^N(Q_{n+1}(1)) + \mu_{n+1}(1) \right]} \right]$$
$$= \gamma_{n+1}^N(1) \; \left[\eta_{n+1}^N(f) - \eta_n^N K_{n+1,(\gamma_n^N(1),\eta_n^N)}(f) \right]$$

and therefore

$$\gamma_{n+1}^N(f) - \mathbb{E}\left(\gamma_{n+1}^N(f) \Big| \mathcal{F}_n^{(N)} \right) = \gamma_{n+1}^N(1) \; \left[\eta_{n+1}^N(f) - \eta_n^N K_{n+1,(\gamma_n^N(1),\eta_n^N)}(f) \right]$$

Finally, we observe that

$$\mathbb{E}\left(\gamma_{n+1}^N(f) \Big| \mathcal{F}_n^{(N)} \right) - \gamma_{n+1}(f) = \gamma_n^N(Q_{n+1}(f)) - \gamma_n(Q_{n+1}(f))$$

from which we find the recursive formula

$$\left[\gamma_{n+1}^N - \gamma_{n+1} \right](f) = \gamma_{n+1}^N(1) \; \left[\eta_{n+1}^N - \eta_n^N K_{n+1,(\gamma_n^N(1),\eta_n^N)} \right](f) + \left[\gamma_n^N - \gamma_n \right](Q_{n+1}(f))$$

The end of the proof of (5.4) is now obtained by a simple induction on the parameter n.

Now, we come to the proof of (5.5). Using the fact that

$$\mathbb{E}\left(\gamma_p^N(1)W_p^N(f^{(1)}) \; \gamma_q^N(1)W_q^N(f^{(2)}) \right) = \mathbb{E}\left(\gamma_p^N(1)\gamma_q^N(1)W_p^N(f^{(1)}) \; \mathbb{E}\left(W_q^N(f^{(2)}) \mid \mathcal{F}_{q-1}^N \right) \right)$$
$$= 0$$

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for any $0 \leq p < q \leq n$, and any $f^{(1)} \in \mathcal{B}(E_p)$, and $f^{(2)} \in \mathcal{B}(E_q)$, we prove that

$$N \, \mathbb{E}\left(\left[\gamma_n^N(1) - \gamma_n(1)\right]^2\right) = \sum_{p=0}^{n} \mathbb{E}\left(\gamma_p^N(1)^2 \, \mathbb{E}\left(W_p^N(Q_{p,n}(1))^2 | \mathcal{F}_{p-1}^N\right)\right)$$

Notice that

$$\frac{1}{\gamma_n(1)^2} = \frac{1}{\gamma_p(1)^2} \, \frac{1}{\eta_p(Q_{p,n}(1))^2} \, \left(\frac{\gamma_p(Q_{p,n}(1))}{\gamma_n(1)}\right)^2 \leq \alpha_{p,n}^\star(\gamma_p(1))^2 \frac{1}{\gamma_p(1)^2} \, \frac{1}{\eta_p(Q_{p,n}(1))^2} \quad (5.6)$$

The r.h.s. estimate comes from the fact that

$$\frac{\gamma_p(Q_{p,n}(1))}{\gamma_n(1)} = \frac{\gamma_p(1) \, \eta_p(Q_{p,n}(1))}{\gamma_p(1) \, \eta_p(Q_{p,n}(1)) + \sum_{p<q\leq n} \mu_q Q_{q,n}(1)} = \alpha_{p,n}\left(\gamma_p(1), \eta_p\right) \leq \alpha_{p,n}^\star(\gamma_p(1))$$

Using the above decompositions, we readily prove that

$$N \, \mathbb{E}\left(\left[\frac{\gamma_n^N(1)}{\gamma_n(1)} - 1\right]^2\right) \leq \sum_{p=0}^{n} \alpha_{p,n}^\star(\gamma_p(1))^2 \, \mathbb{E}\left(\left(\frac{\gamma_p^N(1)}{\gamma_p(1)}\right)^2 \, \mathbb{E}\left(W_p^N(\overline{Q}_{p,n}(1))^2 | \mathcal{F}_{p-1}^N\right)\right)$$

with

$$\overline{Q}_{p,n}(1) = \overline{Q}_{p,n}(1)/\eta_p(Q_{p,n}(1)) \leq q_{p,n}$$

We set

$$U_n^N := \mathbb{E}\left(\left[\frac{\gamma_n^N(1)}{\gamma_n(1)} - 1\right]^2\right) \quad \text{then we find that} \quad N \, U_n^N \leq a_n + \sum_{p=0}^{n} b_{p,n} \, U_p^N$$

with the parameters

$$a_n := \sum_{p=0}^{n} \left(q_{p,n} \alpha_{p,n}^\star(\gamma_p(1))\right)^2 \quad \text{and} \quad b_{p,n} := \left(q_{p,n} \alpha_{p,n}^\star(\gamma_p(1))\right)^2$$

Using the fact that $b_{n,n} \leq 1$, we prove the following recursive equation

$$U_n^N \leq a_n^N + \sum_{0 \leq p < n} b_{p,n}^N \, U_p^N \quad \text{with} \quad a_n^N := \frac{a_n}{N-1} \quad \text{and} \quad b_{p,n}^N := \frac{b_{p,n}}{N-1}$$

Using an elementary proof by induction on the time horizon n, we prove the following inequality:

$$U_n^N \leq \left[\sum_{p=1}^{n} a_p^N \sum_{e \in \langle p,n \rangle} b^N(e)\right] + \left[\sum_{e \in \langle 0,n \rangle} b^N(e)\right] U_0^N$$

In the above display, $\langle p, n \rangle$ stands for the set of all integer valued paths $e = (e(l))_{0 \leq l \leq k}$ of a given length k from p to n

$$e_0 = p < e_1 < \ldots < e_{k-1} < e_k = n \quad \text{and} \quad b^N(e) = \prod_{1 \leq l \leq k} b_{e(l-1),e(l)}^N$$

We have also used the convention $b^N(\emptyset) = \prod_\emptyset = 1$ and $\langle n, n \rangle = \{\emptyset\}$, for $p = n$. Recalling that $\gamma_0^N = \gamma_0$, we conclude that

$$U_n^N \leq \sum_{p=1}^{n} a_p^N \sum_{e \in \langle p,n \rangle} b^N(e)$$

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We further assume that the mixing condition $(M)_k$ presented in (4.1) is met for some parameters $k \geq 1$, and some constant parameters $\epsilon_p = \epsilon > 0$. In this case, we use the fact that

$$\alpha_{p,n}^\star \leq 1 \quad \text{and} \quad q_{p,n} \leq \delta_k/\epsilon$$

to prove that

$$\sup_{0 \leq p \leq n} a_p^N \leq (n+1)\,(\delta_k/\epsilon)^2/(N-1) \quad \text{and} \quad \sup_{0 \leq p \leq n} b_{p,n}^N \leq (\delta_k/\epsilon)^2/(N-1)$$

Using these rather crude estimates, we find that

$$U_n^N \leq a_n^N + \sum_{0 < p < n} a_p^N \sum_{l=1}^{(n-p)} \binom{n-p-1}{l-1} \left(\frac{\delta_k^2}{\epsilon^2(N-1)}\right)^l$$

and therefore

$$U_n^N \leq \frac{(n+1)}{(N-1)} \frac{\delta_k^2}{\epsilon^2} \left(1 + \frac{\delta_k^2}{\epsilon^2(N-1)} \sum_{0 < p < n} \left(1 + \left(\frac{\delta_k^2}{\epsilon^2(N-1)}\right)\right)^{n-p-1}\right)$$

$$= \frac{(n+1)}{(N-1)} \frac{\delta_k^2}{\epsilon^2} \left(1 + \frac{\delta_k^2}{\epsilon^2(N-1)}\right)^{n-1}$$

This ends the proof of the theorem. ∎

5.2.2 Probability distributions

This section is mainly concerned with the proof of the $\mathbb{L}_r$-mean error estimates stated in (3.9). We use the decomposition

$$\left(\gamma_n^N(1), \eta_n^N\right) - \left(\gamma_n(1), \eta_n\right) = \left[\Gamma_{0,n}\left(\gamma_0^N(1), \eta_0^N\right) - \Gamma_{0,n}\left(\gamma_0(1), \eta_0\right)\right]$$

$$+ \sum_{p=1}^{n} \left[\Gamma_{p,n}\left(\gamma_p^N(1), \eta_p^N\right) - \Gamma_{p-1,n}\left(\gamma_{p-1}^N(1), \eta_{p-1}^N\right)\right] \qquad (5.7)$$

to prove that

$$\eta_n^N - \eta_n$$

$$= \left[\Gamma_{0,n}^2\left(\gamma_0^N(1), \eta_0^N\right) - \Gamma_{0,n}^2\left(\gamma_0(1), \eta_0\right)\right] + \sum_{p=1}^{n} \left[\Gamma_{p,n}^2\left(\gamma_p^N(1), \eta_p^N\right) - \Gamma_{p-1,n}^2\left(\gamma_{p-1}^N(1), \eta_{p-1}^N\right)\right]$$

Using the fact that

$$\Gamma_{p-1,n}(m, \eta) = \Gamma_{p,n}\left(\Gamma_p(m, \eta)\right) \Rightarrow \Gamma_{p-1,n}^2(m, \eta) = \Gamma_{p,n}^2\left(\Gamma_p(m, \eta)\right)$$

we readily check that

$$\Gamma_p\left(\gamma_{p-1}^N(1), \eta_{p-1}^N\right) = \left(\gamma_{p-1}^N(1)\eta_{p-1}^N(G_{p-1}) + \mu_p(1), \Psi_{G_{p-1}}\left(\eta_{p-1}^N\right) M_{p,(\gamma_{p-1}^N(1),\eta_{p-1}^N)}\right)$$

$$= \left(\gamma_p^N(1), \eta_{p-1}^N K_{p,(\gamma_{p-1}^N(1),\eta_{p-1}^N)}\right)$$

Since we have $\gamma_0^N(1) = \mu_0(1) = \gamma_0(1)$, one concludes that

$$\eta_n^N - \eta_n = \left[\Gamma_{0,n}^2\left(\gamma_0(1), \eta_0^N\right) - \Gamma_{0,n}^2\left(\gamma_0(1), \eta_0\right)\right]$$
$$+ \sum_{p=1}^n \left[\Gamma_{p,n}^2\left(\gamma_p^N(1), \eta_p^N\right) - \Gamma_{p,n}^2\left(\gamma_p^N(1), \eta_{p-1}^N K_{p,(\gamma_{p-1}^N(1), \eta_{p-1}^N)}\right)\right]$$

Using the fact that $\gamma_p^N(1) \in I_p$, for any $p \geq 0$, the end of the proof is a direct consequence of lemma 4.5 and Kintchine inequality. The proof of the uniform convergence estimates stated in the end of theorem 3.2 are a more or less direct consequence of the functional inequalities derived at the end of section 4.3. The end of the proof of the theorem 3.2 is now completed.

We end this section with the fluctuations properties of the N-particle approximation measures γ_n^N and η_n^N around their limiting values. Using the type of arguments as those used in the proof of the functional central limit theorem, theorem 3.3 in [6], we can prove that the sequence $(W_n^N)_{n \geq 0}$ defined in (3.12) converges in law, as N tends to infinity, to the sequence of n independent, Gaussian and centered random fields $(W_n)_{n \geq 0}$ with a covariance function given in (3.13). Using the decompositions (5.4) and

$$\eta_n^N(f) - \eta_n(f) = \frac{\gamma_n(1)}{\gamma_n^N(1)}\left([\gamma_n^N - \gamma_n]\left(\frac{1}{\gamma_n(1)}(f - \eta_n(f))\right)\right)$$

by the continuous mapping theorem, we deduce the functional central limit theorem 3.3.

6   Particle approximations of spontaneous birth measures

Assume that the spontaneous birth measures μ_n are chosen so that $\mu_n \ll \lambda_n$ for some reference probability measures λ_n and that the Radon Nikodim derivatives $H_n = d\mu_n/d\lambda_n$ are bounded. For any $n \geq 0$, we let $\lambda_n^{N'} := \frac{1}{N'}\sum_{i=1}^{N'}\delta_{\zeta_n^i}$ be the empirical measure associated with N' independent and identically distributed random variables $\left(\zeta_n^i\right)_{1 \leq i \leq N}$ with common distribution λ_n. We also denote by $\mu_n^{N'}$ the particle spontaneous birth measures defined below

$$\forall n \geq 0 \qquad \mu_n^{N'}(dx_n) := H_n(x_n)\,\lambda_n^{N'}(dx_n)$$

In this notation, the initial distribution η_0 and the initial mass γ_0 are approximated by the weighted occupation measure $\eta_0^{N'} := \Psi_{H_0}(\lambda_0^{N'})$ and $\gamma_0^{N'}(1) := \lambda_0^{N'}(H_0)$.

We let $\widetilde{\gamma}_n^{N'}$ and $\widetilde{\eta}_n^{N'}$ the random measures defined as γ_n and η_n by replacing in (2.2) the measures μ_n by the random measures $\mu_n^{N'}$, for any $n \geq 0$; that is, we have that

$$\widetilde{\gamma}_n^{N'} = \widetilde{\gamma}_{n-1}^{N'}Q_n + \mu_n^{N'} \quad \text{and} \quad \widetilde{\eta}_n^{N'}(f_n) = \widetilde{\gamma}_n^{N'}(f_n)/\widetilde{\gamma}_n^{N'}(1)$$

for any $f_n \in \mathcal{B}(E_n)$. By construction, using the same arguments as the ones we used in the proof of (4.8), we have

$$\widetilde{\gamma}_n^{N'} = \sum_{0 \leq p \leq n} \mu_p^{N'}Q_{p,n}$$

This yields for any $f \in \mathcal{B}(E_n)$ the decomposition

$$\left[\widetilde{\gamma}_n^{N'} - \gamma_n\right](f) = \sum_{0 \leq p \leq n}\left[\mu_p^{N'} - \mu_p\right]Q_{p,n}(f) = \sum_{0 \leq p \leq n}\left[\lambda_p^{N'} - \lambda_p\right](H_p\,Q_{p,n}(f))$$

Several estimates can be derived from these formulae, including $\mathbb{L}_p$-mean error bounds, functional central limit theorems, empirical process convergence, as well as sharp exponential concentration inequalities. For instance, we have the unbiasedness property

$$\mathbb{E}\left(\widetilde{\gamma}_n^{N'}(f)\right) = \gamma_n(f)$$

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and the variance estimate

$$N \, \mathbb{E}\left(\left[\widetilde{\gamma}_n^{N'}(f) - \gamma_n(f) \right]^2 \right) = \sum_{0 \le p \le n} \lambda_p \left[(H_p Q_{p,n}(f) - \lambda_p (H_p Q_{p,n}(f)))]^2 \right)$$

Using the same arguments as the ones we used in (5.6), we prove the following rather crude upper bound

$$N \, \mathbb{E}\left(\left[\frac{\widetilde{\gamma}_n^{N'}(f)}{\gamma_n(1)} - \eta_n(f) \right]^2 \right) \le \sum_{0 \le p \le n} \alpha_{p,n}^\star (\gamma_p(1))^2 \frac{1}{\gamma_p(1)^2} \frac{\mu_p \left(H_p Q_{p,n}(f)^2 \right)}{\eta_p (Q_{p,n}(1))^2}$$

$$\le \sum_{0 \le p \le n} \alpha_{p,n}^\star (\gamma_p(1))^2 \frac{1}{\gamma_p(1)^2} \, \|H_p\| \, \mu_p(1) \, q_{p,n}^2$$

We illustrate these variance estimates for time homogeneous models $(E_n, G_n, H_n, M_n, \mu_n) = (E, G, H, M, \mu)$, in the three situations discussed in (4.13), (4.14), and (4.16). We further assume that the mixing condition $(M)_k$ presented in (4.1) is met for some parameters $k \ge 1$, and some $\epsilon > 0$. In this case, we use the fact that $q_{p,n} \le \delta_k / \epsilon$, to prove that

$$N \, \mathbb{E}\left(\left[\frac{\widetilde{\gamma}_n^{N'}(f)}{\gamma_n(1)} - \eta_n(f) \right]^2 \right) \le c \sum_{0 \le p \le n} \left[\alpha_{p,n}^\star (\gamma_p(1)) / \gamma_p(1) \right]^2$$

with some constant $c := \left(\|H\| \, \mu(1) \, (\delta_k / \epsilon)^2 \right)$.

1. When $G(x) = 1$ for any $x \in E$, we have $\gamma_p(1) = \gamma_0(1) + \mu(1) \, p$. Recalling that $\alpha_{p,n}^\star (\gamma_p(1)) \le 1$, we prove the uniform estimates

$$N \, \sup_{n \ge 0} \mathbb{E}\left(\left[\frac{\widetilde{\gamma}_n^{N'}(f)}{\gamma_n(1)} - \eta_n(f) \right]^2 \right) \le c \sum_{p \ge 0} (\gamma_0(1) + \mu(1) \, p)^{-2}$$

2. When $g_+ < 1$ and when the mixing condition $(M)_k$ stated in (4.1) is satisfied, we have seen in (4.15) that

$$\alpha_{p,n}^\star (\gamma_p(1)) \le 1 \wedge \left(d_1 \, g_+^{(n-p)} \right) \quad \text{and} \quad \inf_n \gamma_n(1) \ge d_2$$

for some finite constants $d_1 < \infty$ and $d_2 > 0$. From previous calculations, we prove the following uniform variance estimates

$$N \, \sup_{n \ge 0} \mathbb{E}\left(\left[\frac{\widetilde{\gamma}_n^{N'}(f)}{\gamma_n(1)} - \eta_n(f) \right]^2 \right) \le (c/d_2^2) \sum_{p \ge 0} \left[1 \wedge \left(d_1^2 \, g_+^{2p} \right) \right]$$

3. When $g_- > 1$ we have seen in (4.16) that $\gamma_n(1) \ge d \, g_-^n$ for any $n \ge n_0$, for some finite constant $d < \infty$ and some $n_0 \ge 1$ so

$$N \, \sup_{n \ge 0} \mathbb{E}\left(\left[\frac{\widetilde{\gamma}_n^{N'}(f)}{\gamma_n(1)} - \eta_n(f) \right]^2 \right) \le c \left(\sum_{0 \le p \le n_0} \gamma_p(1)^{-2} + d \sum_{n \ge n_0} g_-^{-2n} \right)$$

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